

# Hausdorff Dimension and Hausdorff Measure for Non-integer based Cantor-type Sets

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## Abstract

We consider digits-deleted sets or Cantor-type sets with  $\beta$ -expansions. We calculate the Hausdorff dimension  $d$  of these sets and show that  $d$  is continuous with respect to  $\beta$ . The  $d$ -dimensional Hausdorff measure of these sets is finite and positive.

## 1 Introduction

The Hausdorff dimension and Hausdorff measure of expansions with deleted digits are of interest to mathematicians. The Cantor middle-third set is a classical example. In 1993, M. Keane posed the following question, see [22]:

*Is the Hausdorff dimension  $d(\lambda)$  of the one parameter family of Cantor-type sets*

$$\Lambda(\lambda) = \left\{ \sum_{k=1}^{\infty} i_k \lambda^k : i_k = 0, 1, 3 \right\}$$

*continuous for  $\lambda \in [\frac{1}{4}, \frac{1}{3}]$ ?*

This question is answered by Pollicott and Simon [22] in 1995. They show that  $d(\lambda) = \frac{\log 3}{-\log \lambda}$  for almost all  $\lambda \in [\frac{1}{4}, \frac{1}{3}]$  and there exists a dense set with  $d(\lambda) < \frac{\log 3}{-\log \lambda}$ . Solomyak [26] in 1998 shows that the  $d(\lambda)$ -dimensional Hausdorff measure of  $\Lambda(\lambda)$  is zero for almost all  $\lambda \in [\frac{1}{4}, \frac{1}{3}]$ . Keane, Smorodinsky and Solomyak study the size of  $\Lambda(\lambda)$  when  $\lambda > \frac{1}{3}$  ([15], 1995).

Notice the definition of  $\Lambda(\lambda)$  implies no restriction on the digits 0, 1 and 3. When  $\lambda > \frac{1}{4}$ , different sequences may express the same number. In this paper we study Hausdorff dimension in a more restricted case, associated with

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$\beta$ -expansions. We obtain very specific new results for digits-deleted sets or Cantor-type sets with  $\beta$ -expansions.

$\beta$ -transformations and  $\beta$ -expansions are first introduced by Rényi [23] in 1957 and further explored by Parry [20] in 1960. For fixed  $\beta > 1$ ,  $T_\beta : [0, 1) \rightarrow [0, 1)$  is defined by

$$T_\beta x = \beta x - \lfloor \beta x \rfloor, \quad (1)$$

where  $\lfloor \cdot \rfloor$  is the floor function. By (1) we can define the  $\beta$ -expansion for any  $x \in [0, 1)$ . Let  $a_1 = \lfloor \beta x \rfloor$  and  $a_n = \lfloor \beta T_\beta^{n-1} x \rfloor$ . Then

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots \quad (2)$$

If we denote  $x = (0.a_1 a_2 \cdots)_\beta$ , then  $T_\beta x = (0.a_2 \cdots)_\beta$ . Rényi [23] and Parry [20] study the ergodic theory for  $T_\beta$ .

Given  $\beta > 2$ , for  $0 \leq \theta_0 < \theta_1 < \cdots < \theta_{q-1} \leq \lfloor \beta \rfloor$ , we define

$$C_{\beta; \theta_0 \cdots \theta_{q-1}} = \overline{\{x = (0.a_1 a_2 \cdots)_\beta : a_k \in \{\theta_0, \theta_1, \dots, \theta_{q-1}\}\}}.$$

Corresponding to  $\Lambda(\lambda)$  defined above, we have  $C_{\beta; 013}$  where  $\beta = \frac{1}{\lambda} \in (3, 4]$ . It is easy to see that  $\Lambda(\lambda) \supseteq C_{\beta; 013}$  with equality only when  $\lambda = \frac{1}{4}$ . The classical Cantor middle-third set is  $C_{3; 02}$ . It is natural to ask the following questions:

*What is the Hausdorff dimension and Hausdorff measure of  $C_{\beta; \theta_0 \cdots \theta_{q-1}}$ ? Is  $\dim_H(C_{\beta; \theta_0 \cdots \theta_{q-1}})$  continuous with respect to  $\beta$ ?*

Comparing  $\Lambda(\lambda)$  and  $C_{\beta; 013}$ , we see that the former is an attractor of the iterated function system (IFS)  $(X; f_0, f_1, f_2)$  where  $X = [0, \frac{3\lambda}{1-\lambda}]$  and

$$f_0(x) = \lambda x, \quad f_1(x) = \lambda(x+1) \quad \text{and} \quad f_2(x) = \lambda(x+3),$$

but when  $\beta$  is not an integer it is impossible to consider  $C_{4; 013}$  as an attractor of an IFS without introducing elaborate additional machineries.

Given an IFS  $(X; f_0, \dots, f_{n-1})$  and  $\beta < n$ , we define the  $\beta$ -attractor as a certain compact subset of the attractor determined by the  $\beta$ -shift. We show that when all the maps are similarities with the same contraction ratio  $0 < r < 1$  and with a separation condition, the Hausdorff dimension of the  $\beta$ -attractor is given by  $s = \frac{\log \beta}{-\log r}$  and the  $s$ -dimensional Hausdorff measure is positive and finite.

For fixed  $\beta > 2$  and the set of digits  $0 \leq \theta_0 < \cdots < \theta_{q-1} \leq \lfloor \beta \rfloor$ , we show that there exists a number  $\alpha > 1$  such that  $C_{\beta; \theta_0 \cdots \theta_{q-1}}$  is an  $\alpha$ -attractor of an IFS. The Hausdorff dimension of  $C_{\beta; \theta_0 \cdots \theta_{q-1}}$  is  $s = \frac{\log \alpha}{\log \beta}$  and the  $s$ -dimensional Hausdorff measure of  $C_{\beta; \theta_0 \cdots \theta_{q-1}}$  is positive and finite. If the separation condition holds we obtain this immediately from the general result for  $\beta$ -attractors. However the separation condition does not hold in general. We need a direct proof. We also show that  $\dim_H(C_{\beta; \theta_0 \cdots \theta_{q-1}})$  is continuous with respect to  $\beta$  for  $\beta > \theta_{q-1}$  and it has a negative derivative with respect to  $\beta$  for almost all  $\beta > \theta_{q-1}$ . There

exists a nowhere dense subset of  $(\theta_{q-1}, \theta_{q-1} + 1]$  with Lebesgue measure 0 such that  $\dim_H(C_{\beta, \theta_0 \dots \theta_{q-1}})$  has infinite derivative or infinite one-sided derivative.

$\beta$ -attractors fit into the more general *symbolic construction* of Pesin and Weiss [21]. A  $\beta$ -attractor is the limit set of a symbolic construction using the  $\beta$ -shift. If a symbolic construction is *regular*, Pesin and Weiss in [21] give a lower bound to the Hausdorff dimension of the limit set using topological pressure. If  $\beta$  is simple then the symbolic construction using the  $\beta$ -shift is regular under a separation condition. It is not known this is true for all  $\beta > 1$ . Barreira [4] gives a sufficient condition for a symbolic construction is regular. However the construction of  $C_{\beta, \theta_0 \dots \theta_{q-1}}$  does not obey this condition for some value of  $\beta$ . If the symbolic construction is regular and the equilibrium measure associated with an  $l$ -estimating vector is a Gibbs measure then [21] shows that the  $s$ -dimensional Hausdorff measure of the limit set of a symbolic construction is positive, where  $s$  is a lower bound of the Hausdorff dimension of the limit set related to the  $l$ -estimating vector. It is known that if a symbolic system has the specification property then any equilibrium measure is Gibbs ([24]). A  $\beta$ -shift has the specification property if and only if the length of strings of 0's in the  $\beta$ -expansion of 1 is bounded ([5]). Obviously this set does not contain any interval. Schmeling [25] showed that this set has Hausdorff dimension 1. Again, it is not known if every  $\beta$ -shift possesses a Gibbs measure as its equilibrium measure associated with the  $l$ -estimating vector of [21].

Although the symbolic construction for  $\beta$ -attractors may not be regular in general, the symbolic construction of  $C_{\beta, \theta_0 \dots \theta_{q-1}}$  is regular. It follows that the Hausdorff dimension of  $C_{\beta, \theta_0 \dots \theta_{q-1}}$  can be obtained from the results of [21] by considering the topological pressure of the corresponding  $\alpha$ -shift. However, as mentioned above, in this way we can not get the information of Hausdorff measure of  $C_{\beta, \theta_0 \dots \theta_{q-1}}$  in all cases. To overcome this difficulty we give a new proof.

It is well known that the classical Cantor set can be defined to be the digit-deleted set,  $C = C_{3;02}$ . It can also be defined geometrically as the attractor of the IFS  $F = \{[0, 1]; f_0, f_2\}$ , where  $f_i(x) = \frac{x+i}{3}$  for  $i = 0, 1, 2$  are the inverse branches of  $T_3$ . But for  $2 < \beta < 3$  the digit-deleted set  $C_{\beta;02}$  is not the attractor of an IFS of inverse branches of  $T_\beta$  because of the piecewise definition of  $T_\beta^{-1}$ , namely

$$T_\beta^{-1}x = \begin{cases} \frac{x}{\beta}, & \frac{x+1}{\beta}, & \frac{x+2}{\beta}, & \text{for } x < \beta - 2 \\ \frac{x}{\beta}, & \frac{x+1}{\beta}, & & \text{for } x \geq \beta - 2. \end{cases}$$

However we will show that there is an associated *local* IFS (see [6], p177) whose invariant sets are related to  $C_{\beta;02}$  in an interesting way. A local IFS has its domain of at least one of its functions not equal to the whole of the underlying space,  $[0, 1]$  in the present case. We define

$$F_\beta = \{[0, 1]; f_0 : [0, 1] \mapsto [0, 1], f_2 : [0, \beta - 2] \mapsto [0, 1]\}$$

where now  $f_0(x) = \frac{x}{\beta}$ , and  $f_2(x) = \frac{x+2}{\beta}$ . An invariant set  $A$  of the local IFS  $F_\beta$

is a nonempty compact subset of  $[0, 1]$  such that

$$A = f_0(A) \cup f_2(A \cap [0, \beta - 2]).$$

In general, a local IFS may have no or many invariant sets ([6]). We show that, with the exception of countably many values of  $\beta$ ,  $C_{\beta;02}$  is the unique invariant set of the local IFS  $F_\beta$ . Otherwise,  $F_\beta$  possesses another invariant set  $B_{\beta;02}$  which can be constructed by an interval removal process. We have  $C_{\beta;02} \subset B_{\beta;02}$  and  $B_{\beta;02} \setminus C_{\beta;02}$  consists of countably many isolated points.

$\beta$ -transformations and  $\beta$ -expansions are of interest to mathematicians in a broad range. After the pioneer works of Rényi [23] and Parry [20], many research works related to  $\beta$ -transformations and  $\beta$ -expansions have been published. Among these works, [1], [2], [8], [11], [13], [14], [17] and [19], for example, have studied fractals or fractal sets related to  $\beta$ -expansions. Barnsley [7] used  $\beta$ -transformation to study fractal tops.

In section 2, we recall concepts and basic results for Hausdorff dimension, iterated function systems,  $\beta$ -expansions and symbolic dynamical systems. In section 3, we study the Hausdorff dimension and Hausdorff measure for  $\beta$ -attractors under a separation condition. In section 4, we study the Hausdorff dimension and Hausdorff measure for  $C_{\beta;\theta_0 \dots \theta_{q-1}}$ , the Cantor-type sets constructed by  $\beta$ -expansions. In section 5, we study the invariant sets of the local IFS  $F_\beta$ . In section 6, we point out some interesting topics for further research.

## 2 Hausdorff Dimension, Iterated Function Systems, $\beta$ -expansion and Symbolic Dynamical Systems

In this section we introduce concepts and definitions for Hausdorff dimension, iterated function systems,  $\beta$ -expansion and symbolic dynamical systems.

Readers can find concepts of Hausdorff measure, Hausdorff dimension and iterated function systems in [9] or [10]. For the original source of iterated function systems see [12].

Let  $E$  be a subset of a metric space  $X$ . A  $\delta$ -cover of  $E$  is a countable or finite collection  $\{U_i\}$  of subsets of  $X$  with  $|U_i| \leq \delta$  such that  $E \subset \cup_{i=1}^{\infty} U_i$ , where  $|\cdot|$  is the diameter of the given set. For any  $\delta > 0$  and  $s \geq 0$ , define

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}.$$

Clearly,  $\mathcal{H}_\delta^s(E)$  is decreasing with respect to either  $s$  or  $\delta$ . Let

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E).$$

We call  $\mathcal{H}^s(E)$  the  $s$ -dimensional Hausdorff out measure of  $E$ . The Hausdorff dimension of  $E$  is defined by

$$\dim_H(E) = \inf\{s \geq 0 : \mathcal{H}^s(E) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(E) = \infty\}.$$

An iterated function system (IFS)  $F = (X; f_0, f_1, \dots, f_{n-1})$  on a compact metric space  $X$  consists of a number of contractions  $f_i : X \rightarrow X$ , where  $n \geq 2$ . There exists a non-empty compact subset  $E$  of  $X$  such that

$$E = \bigcup_{i=0}^{n-1} f_i(E).$$

We say  $E$  is the attractor of the IFS. For any sequence  $i_1, i_2, \dots$  with  $0 \leq i_k \leq n-1$ , and any  $x \in X$ , we have

$$\lim_{k \rightarrow \infty} f_{i_1} \circ \dots \circ f_{i_k}(x) \in E.$$

The above limit exists and is independent of  $x \in X$ . On the other hand, for any  $y \in E$  there exists a sequence  $(i_1, i_2, \dots)$  such that

$$y = \lim_{k \rightarrow \infty} f_{i_1} \circ \dots \circ f_{i_k}(x)$$

for any  $x \in X$ . We say  $(i_1, i_2, \dots)$  is an address of  $y$ . One point in  $E$  may have more than one addresses.

We say the Open Set Condition holds for an IFS, if there exists a non-empty open set  $O$  such that  $f_i(O) \subset O$  and  $f_i(O) \cap f_j(O) = \emptyset$  for all  $0 \leq i, j \leq n-1$  and  $i \neq j$ .

Assume that  $X$  is a compact subset of  $\mathbf{R}^m$ . If  $f_i$  is a similarity for all  $i$ , i.e., for all  $x, y \in X$ ,  $d(f_i(x), f_i(y)) = r_i d(x, y)$  for some  $r_i < 1$ , and the Open Set Condition holds, then the Hausdorff dimension of the attractor  $E$  is given by

$$\dim_H(E) = s$$

where  $s$  satisfies  $r_0^s + r_1^s + \dots + r_{n-1}^s = 1$ .

Let  $\Sigma_n = \{0, 1, \dots, n-1\}^{\mathbb{N}}$ . Then  $\Sigma_n$  is compact with respect to the product topology. Define a shift map  $\sigma : \Sigma_n \rightarrow \Sigma_n$  by

$$\sigma(i_1, i_2, \dots) = (i_2, i_3, \dots).$$

We call  $\Sigma_n$  a full shift. A compact subset  $\Sigma$  of  $\Sigma_n$  is said to be a subshift if  $\sigma(\Sigma) = \Sigma$ . A block  $(i_1, \dots, i_k)$  is a forbidden word of  $\Sigma$  if it does not appear in any element of  $\Sigma$ .  $\Sigma$  is a subshift of finite type if it is determined by a finite set of forbidden words (see [16]). Given an  $n \times n$  0-1 matrix  $M$ , let  $\Sigma_M$  contain all the elements  $(i_1, i_2, \dots) \in \Sigma_n$  such that  $m_{i_k+1, i_{k+1}+1} = 1$  for all  $k \geq 1$ . We say  $\Sigma_M$  is a Markov shift. Clearly, a Markov shift is a subshift of finite type.

Fix  $\beta > 1$ . Assume that the  $\beta$ -expansion of  $\beta - \lfloor \beta \rfloor$  is

$$\beta - \lfloor \beta \rfloor = \frac{\epsilon_2}{\beta} + \frac{\epsilon_3}{\beta^2} + \cdots$$

Let  $\epsilon_1 = \lfloor \beta \rfloor$ . Then

$$1 = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \frac{\epsilon_3}{\beta^3} + \cdots \quad (3)$$

We say that (3) is the  $\beta$ -expansion of 1 and denote by  $1 = (0.\epsilon_1\epsilon_2\cdots)_\beta$ . We say  $\beta$  is simple if the  $\beta$ -expansion of 1 has finite many non-zero terms.

Let  $e_i = \epsilon_i$  if  $\beta$  is non-simple; or let  $e_{kn+i} = \epsilon_i$  and  $e_{(k+1)n} = \epsilon_n - 1$ , for all  $k \geq 0$  and  $i = 1, 2, \dots, n-1$ , if  $\beta$  is simple with  $\epsilon_n > 0$  and  $\epsilon_j = 0$  when  $j > n$ . Given a sequence  $(a_1, a_2, \dots)$  with  $a_i \in \{0, 1, \dots, \lfloor \beta \rfloor\}$ , the expression

$$\frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots \quad (4)$$

is the  $\beta$ -expansion of some  $x \in [0, 1)$  if and only if for any  $n \geq 1$  one has

$$(a_n, a_{n+1}, \dots) < (e_1, e_2, \dots),$$

where “ $<$ ” is according to the lexicographical order. When (4) is the  $\beta$ -expansion for some  $x \in [0, 1)$ , we say the finite sequence  $(a_1, a_2, \dots, a_n)$  is  $\beta$ -admissible.

Let

$$1 = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \frac{\epsilon_3}{\beta^3} + \cdots$$

and

$$1 = \frac{\epsilon'_1}{\alpha} + \frac{\epsilon'_2}{\alpha^2} + \frac{\epsilon'_3}{\alpha^3} + \cdots$$

be the  $\beta$  and  $\alpha$ -expansions of 1. If  $\beta < \alpha$  then

$$(\epsilon_1, \epsilon_2, \dots) < (\epsilon'_1, \epsilon'_2, \dots).$$

Use  $\Sigma_\beta$  to denote the closure of all  $\beta$ -admissible sequences under product topology. Then  $\Sigma_\beta$  is a subshift which we call the  $\beta$ -shift. If  $\beta = n$  is an integer, then  $\Sigma_\beta$  is the full shift. If  $\beta$  is simple, then  $\Sigma_\beta$  a subshift of finite type. For example, when  $\beta = \frac{\sqrt{5}+1}{2}$ , the golden mean,  $\Sigma_\beta$  is a Markov shift with  $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ .

### 3 $\beta$ -attractors and Their Hausdorff Dimension.

Let  $(X; f_0, f_1, \dots, f_{n-1})$  be an IFS with attractor  $E$ . Define  $\phi : \Sigma_n \mapsto E$  by

$$\phi(i_1, i_2, \dots) = \lim_{k \rightarrow \infty} f_{i_1} \circ \cdots \circ f_{i_k}(x).$$

$\phi$  is well defined since the limit in the right hand side exists, is independent of  $x$ , and is in  $E$ .

**Definition.** Given an IFS  $(X; f_0, f_1, \dots, f_{n-1})$  with attractor  $E$ . For a subshift  $\Sigma$  of  $\Sigma_n$  let

$$E_\Sigma = \phi(\Sigma).$$

We call  $E_\Sigma$  the  $\Sigma$ -attractor of the given IFS. In particular, when  $\Sigma$  is a  $\beta$ -shift for some  $\beta \leq n$ , we use  $E_\beta$  to denote  $E_{\Sigma_\beta}$  and call it the  $\beta$ -attractor.

**Remark.** The  $\Sigma$ -attractor defined above is a special case of the more general symbolic construction of [21]. Many different settings of fractals with iterated function systems can be viewed as  $\Sigma$ -attractors for some subshifts. For example, the fractals defined by Markov shifts (see [3] or [27]), the graph-directed fractals first proposed by Mauldin and Williams [18] can fit into this setting. However, when  $\beta$  is non-simple, it is difficult to fit  $E_\beta$  into known settings other than the symbolic construction of [21].

Recall that when all  $f_i$ 's are similarities with scale  $r_i < 1$  and with the open set condition, the Hausdorff dimension of the attractor  $E$  is determined by  $\sum_{i=0}^{n-1} r_i^s = 1$ . When all the  $r_i$ 's are equal ( $= r$ , say), we have  $s = \frac{\log n}{-\log r}$ . For  $\beta$ -attractors we have the following result.

**Theorem 1.** *Let  $(X; f_0, f_1, \dots, f_{n-1})$  be an IFS such that*

$$d(f_i(x), f_i(y)) = rd(x, y)$$

*for all  $x, y \in X$  and  $0 \leq i \leq n-1$ , where  $0 < r < 1$ . Let  $1 < \beta < n$ . Assume that the  $\beta$ -attractor  $E_\beta$  has the following separation condition:*

$$f_i(E_\beta) \cap f_j(E_\beta) \cap E_\beta = \emptyset, \text{ for } i \neq j.$$

*Then the Hausdorff dimension of  $E_\beta$  is given by*

$$\dim(E_\beta) = \frac{\log \beta}{-\log r}.$$

*The  $s$ -dimensional Hausdorff measure of  $E_\beta$  is positive and finite, where  $s = \dim_H(E_\beta)$ .*

Obviously, theorem 1 holds when  $\beta$  is an integer. Comparing the integer case, we may interpret the  $\beta$ -attractor as it is constructed by  $\beta$  many similarities. For some value of  $\beta$ ,  $\dim_H(E_\beta)$  can be computed by existing method. For example, if  $\beta = \frac{\sqrt{5}+1}{2}$ , then  $\Sigma_\beta$  is a Markov shift with  $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  (the golden-mean shift). With the separation condition, the Hausdorff dimension can be calculated by  $\|MR^s\| = 1$ , where  $MR^s = \begin{pmatrix} r^s & r^s \\ r^s & 0 \end{pmatrix}$  and  $\|\cdot\|$  is the Perron-Frobenius eigenvalue of the given matrix. Then we have  $1 - r^s - r^{2s} = 0$ . This gives  $r^s = \beta^{-1}$  and  $s = \frac{\log \beta}{-\log r}$ . If  $\beta$  is given by  $1 = \frac{1}{\beta} + \frac{1}{\beta^3}$ , then the

$\beta$ -expansion of real numbers induce a shift of finite type with forbidden words determined by the set  $\{(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$ . Define an IFS formed by the composed maps,  $\{f_0 \circ f_0, f_0 \circ f_1, f_1 \circ f_0\}$ . Then  $E_\beta$  is the Markov attractor with

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Noting that the composed maps are similarities with scales  $r^2$ , then with the separation condition, the Hausdorff dimension of  $E_\beta$  is given by  $\|MR^{2s}\| = 1$ . This gives us

$$r^{6s} + 2r^{4s} + r^{2s} - 1 = 0.$$

But

$$\lambda^6 + 2\lambda^4 + \lambda^2 - 1 = (\lambda^3 + \lambda - 1)(\lambda^3 + \lambda + 1).$$

Hence we have  $r^{3s} + r^s = 1$  which implies  $r^s = \beta^{-1}$ , and therefore  $s = \frac{\log \beta}{-\log r}$ .

Obviously, this discussion can only apply to particular values of  $\beta$  when the set of forbidden words are “short” and the transition matrix is “small” of size whose eigenvalue is “calculatable”. It is hard to apply to general case. Besides, it is not applicable when the related shift is not a subshift of finite type.

Use  $\mathcal{S}_\beta^k$  to denote the set of all  $\beta$ -admissible sequences of length  $k$ . To prove Theorem 1 we need to estimate the size of  $\mathcal{S}_\beta^k$ . The following result can be found in [23] (equations (4.9), (4.10)).

**Lemma 1.** *We have*

$$\beta^k \leq |\mathcal{S}_\beta^k| \leq \frac{\beta^{k+1}}{\beta - 1}.$$

*Proof of Theorem 1.* First we show that  $\dim_H(E_\beta) \leq \frac{\log \beta}{-\log r}$ . For  $(i_1, \dots, i_k) \in \mathcal{S}_\beta^k$ , denote  $\Delta_{i_1 \dots i_k} = f_{i_1} \circ \dots \circ f_{i_k}(X)$ . Then the collection  $\{\Delta_{i_1 \dots i_k} : (i_1, \dots, i_k) \in \mathcal{S}_\beta^k\}$  is a cover of  $E_\beta$ . Since all  $f_i$ 's are similarities with scale  $r$ , we have  $|\Delta_{i_1 \dots i_k}| \leq r^k |X|$ . By Lemma 1, we get

$$\sum_{i_1 \dots i_k \in \mathcal{S}_\beta^k} |\Delta_{i_1 \dots i_k}|^s \leq \frac{\beta^{k+1}}{\beta - 1} r^{sk} |E_\beta|^s = \frac{\beta}{\beta - 1} |X|^s,$$

where  $s = \frac{\log \beta}{-\log r}$ . This shows that  $\dim(E_\beta) \leq s$  and the  $s$ -dimensional Hausdorff measure of  $E_\beta$  is finite.

Next we show that  $\dim(E_\beta) \geq \frac{\log \beta}{-\log r}$ . We will show that for some  $\delta_0 > 0$  we have  $\mathcal{H}_{\delta_0}^s(E_\beta) > 0$  where  $s = \frac{\log \beta}{-\log r}$ . Then  $\mathcal{H}^s(E_\beta) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E_\beta) \geq \mathcal{H}_{\delta_0}^s(E_\beta)$ . This gives  $\dim_H(E_\beta) \geq s$  and the  $s$ -dimensional Hausdorff measure of  $E_\beta$  is positive.



By the separation condition,

$$\delta_0 = \min_{i \neq j} \{d(f_i(E_\beta) \cap E_\beta, f_j(E_\beta) \cap E_\beta)\} > 0.$$

Then there exists  $l > 0$  such that  $r^{l+1}|X| < \delta_0 \leq r^l|X|$ . Let  $\mathcal{U} = \{U_i\}$  be a  $\delta_0$  cover of  $E_\beta$ . Since  $E_\beta$  is compact without loss of generality we may assume that  $\mathcal{U} = \{U_1, \dots, U_N\}$  is a finite cover. We may also assume that  $U_i \subset E_\beta$ . In fact we can use  $U_i \cap E_\beta$  to replace  $U_i$ . Choose  $k_i$  such that

$$r^{k_i+1}|X| < |U_i| \leq r^{k_i}|X|.$$

We claim that  $U_i \subset \Delta_{i_1 \dots i_{k_i-l}}$  for some  $\beta$ -admissible  $(i_1, \dots, i_{k_i-l})$ . Taking  $y \in U_i$  which has an address  $(i_1, i_2, \dots) \in \Sigma_\beta$ , for any  $z \in E_\beta$  with an address  $(j_1, j_2, \dots) \in \Sigma_\beta$  such that  $(i_1, \dots, i_{k_i-l}) \neq (j_1, \dots, j_{k_i-l})$ . Assume that  $i_1 = j_1, \dots, i_p = j_p$  but  $i_{p+1} \neq j_{p+1}$  for some  $p < k_i - l$ . Then

$$d(y, z) = d(f_{i_1} \circ \dots \circ f_{i_p} \circ f_{i_{p+1}}(y_1), f_{i_1} \circ \dots \circ f_{i_p} \circ f_{j_{p+1}}(z_1))$$

for some  $y_1, z_1 \in E_\beta$ . It is obvious that  $f_{i_{p+1}}(y_1), f_{j_{p+1}}(z_1) \in E_\beta$ . Then,

$$\begin{aligned} d(y, z) &= r^p d(f_{i_{p+1}}(y_1), f_{j_{p+1}}(z_1)) \\ &\geq r^p d(f_{i_{p+1}}(E_\beta) \cap E_\beta, f_{j_{p+1}}(E_\beta) \cap E_\beta) \geq r^p \delta_0 \\ &> r^p r^{l+1}|X| \geq r^{k_i-l-1} r^{l+1}|X| \\ &= r^{k_i}|X| > |U_i|. \end{aligned}$$

Hence  $z \notin U_i$  and  $U_i \subset \Delta_{i_1 \dots i_{k_i-l}}$ . Now we get another cover of  $E_\beta$ :

$$\mathcal{C} = \{\Delta_{i_1 \dots i_{k_i-l}} : U_i \subset \Delta_{i_1 \dots i_{k_i-l}}, r^{k_i+1}|X| < |U_i| \leq r^{k_i}|X|, U_i \in \mathcal{U}\}.$$

For  $s > 0$  we have

$$\sum_{\Delta_{i_1 \dots i_{k_i-l}} \in \mathcal{C}} |\Delta_{i_1 \dots i_{k_i-l}}|^s = \sum_{\Delta_{i_1 \dots i_{k_i-l}} \in \mathcal{C}} r^{s(k_i-l)} |X|^s \leq r^{s(-l-1)} \sum_{U_i \in \mathcal{U}} |U_i|^s \quad (5)$$

Let

$$k = \max_{U_i \in \mathcal{U}} \{k_i - l : r^{k_i+1}|X| < |U_i| \leq r^{k_i}|X|\}.$$

Refine  $\mathcal{C}$  into

$$\mathcal{C}' = \{\Delta_{i_1 \dots i_k} : (i_1, \dots, i_k) \in \mathcal{S}_\beta^k\}.$$

Then  $|\Delta_{i_1 \dots i_k}|^s = r^{s(k-k_i)} |\Delta_{i_1 \dots i_{k_i-l}}|^s$ . Set  $s = \frac{\log \beta}{-\log r}$ , then

$$|\Delta_{i_1 \dots i_k}|^s = \beta^{-(k-k_i)} |\Delta_{i_1 \dots i_{k_i-l}}|^s.$$

By Lemma 1 we see that each  $\Delta_{i_1 \dots i_{k_i-l}} \in \mathcal{C}$  contains at most  $\frac{\beta^{k-k_i+l+1}}{\beta-1}$  many  $\Delta_{i_1 \dots i_k} \in \mathcal{C}'$ . Then

$$\sum_{\substack{(j_1, \dots, j_k) \in \mathcal{S}_\beta^k \\ j_1 = i_1, \dots, j_{k_i-l} = i_{k_i-l}}} |\Delta_{j_1 \dots j_k}|^s \leq \frac{\beta^{l+1}}{\beta-1} |\Delta_{i_1 \dots i_{k_i-l}}|^s.$$

Therefore,

$$\sum_{\Delta_{i_1 \dots i_{k_i-l}} \in \mathcal{C}} |\Delta_{i_1 \dots i_{k_i-l}}|^s \geq \beta^{-l-1}(\beta-1) \sum_{\Delta_{i_1 \dots i_k} \in \mathcal{C}'} |\Delta_{i_1 \dots i_k}|^s$$

The right hand side contains all the  $\beta$ -admissible sequences in  $\mathcal{S}_\beta^k$  and we have  $|\Delta_{i_1 \dots i_k}| = r^k |X|$ . Hence

$$\begin{aligned} \sum_{\Delta_{i_1 \dots i_{k_i-l}} \in \mathcal{C}} |\Delta_{i_1 \dots i_{k_i-l}}|^s &\geq \beta^{-l-1}(\beta-1) \sum_{(i_1, \dots, i_k) \in \mathcal{S}_\beta^k} r^{sk} |X|^s \\ &\geq \beta^{-l-1}(\beta-1) \beta^k r^{sk} |X|^s = \beta^{-l-1}(\beta-1) |X|^s. \end{aligned} \quad (6)$$

By (5) and (6) we obtain

$$\begin{aligned} \sum_{U_i \in \mathcal{U}} |U_i|^s &\geq r^{s(l+1)} \sum_{\Delta_{i_1 \dots i_{k_i-l}} \in \mathcal{C}} |\Delta_{i_1 \dots i_{k_i-l}}|^s \\ &\geq r^{s(l+1)} \beta^{-l-1}(\beta-1) |X|^s = \beta^{-2(l+1)}(\beta-1) |X|^s \end{aligned}$$

for any finite  $\delta_0$ -cover  $\mathcal{U}$ . Hence  $\dim_H(E_\beta) \geq s$  and  $\mathcal{H}^s(E_\beta) > 0$ .  $\square$

## 4 Hausdorff Dimension for $C_{\beta; \theta_0 \dots \theta_{q-1}}$

Let  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  be as defined in section 1. Use  $z_\beta$  to denote the maximum of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$ . We have the following result.

**Lemma 2.**  $z_\beta$  can be expressed as

$$z_\beta = \frac{z_1}{\beta} + \frac{z_2}{\beta^2} + \dots \quad (7)$$

where  $z_i \in \{\theta_0, \dots, \theta_{q-1}\}$ , and

$$(z_i, z_{i+1}, \dots) \leq (z_1, z_2, \dots). \quad (8)$$

*Proof.* By the definition of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$ , we have two possibilities: the digits of the  $\beta$ -expansion of  $z_\beta$  are in  $\{\theta_0, \dots, \theta_{q-1}\}$ , or  $z_\beta$  is a limit of numbers whose  $\beta$ -expansion consists of digits in  $\{\theta_0, \dots, \theta_{q-1}\}$ . For the first case, the  $\beta$ -expansion of  $z_\beta$  satisfies the requirement of Lemma 2. For the second case, there exists a sequence  $\{y_k\}$  with digits in  $\{\theta_0, \dots, \theta_{q-1}\}$  such that  $y_k \uparrow z_\beta$ . Assume that  $y_n = (0.y_1^k y_2^k \dots)_\beta$ . Then we have

$$(y_1^k, y_2^k, \dots) < (y_1^{k+1}, y_2^{k+1}, \dots).$$

Since  $\Sigma_\beta$  is compact under the product topology, the sequence  $\{(y_1^k, y_2^k, \dots)\}$  has a limit  $(z_1, z_2, \dots)$ . It is obvious that  $(z_1, z_2, \dots)$  satisfies (7). We show

that it satisfies (8) as well. If it is not true, then  $(z_k, z_{k+1}, \dots) > (z_1, z_2, \dots)$ . Let  $\tilde{y}_n = (0, y_k^n y_{k+1}^n \dots)_\beta$ . Then

$$\lim_{n \rightarrow \infty} \tilde{y}_n = \frac{z_k}{\beta} + \frac{z_{k+1}}{\beta^2} + \dots > z_\beta,$$

a contradiction.  $\square$

Using  $(z_1, z_2, \dots)$  we define a new sequence  $(\omega_1, \omega_2, \dots)$  with  $\omega_i = j$  if  $z_i = \theta_j$ . Then

$$(\omega_k, \omega_{k+1}, \dots) \leq (\omega_1, \omega_2, \dots).$$

Let  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}}$  be determined by

$$1 = \frac{\omega_1}{\alpha} + \frac{\omega_2}{\alpha^2} + \dots \quad (9)$$

Then either (9) is the  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}}$ -expansion of 1 or  $(\omega_1, \omega_2, \dots)$  is periodic.

**Theorem 2.** *The Hausdorff dimension of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is given by*

$$\dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}}) = \frac{\log \alpha_{\beta; \theta_0 \dots \theta_{q-1}}}{\log \beta}.$$

*The  $s$ -dimensional Hausdorff measure of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is positive and finite where  $s = \dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}})$ .*

*Proof.* For  $0 \leq i \leq q-1$  define  $f_i : \left[0, \frac{\beta}{\beta-1}\right] \mapsto \left[0, \frac{\beta}{\beta-1}\right]$  by  $f_i(x) = \frac{x+\theta_i}{\beta}$ . We will show that  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is the  $\alpha$ -attractor of the IFS  $\left(\left[0, \frac{\beta}{\beta-1}\right]; f_0, \dots, f_{q-1}\right)$ . Here and rest of the proof we use  $\alpha$  to denote  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}}$ . Then by the proof of Theorem 1 we get that

$$\dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}}) \leq \frac{\log \alpha}{\log \beta}$$

and the  $s$ -dimensional Hausdorff measure of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is finite for  $s = \frac{\log \alpha}{\log \beta}$ . If we further have the separation condition, then Theorem 2 is proved.

First we show that  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is an  $\alpha$ -attractor. Use  $\Sigma_{\beta; \theta_0 \dots \theta_{q-1}}$  to denote the closure of all sequences  $(a_1, a_2, \dots)$  with  $a_i \in \{\theta_0, \dots, \theta_{q-1}\}$  which forms the  $\beta$ -expansion of some  $x \in [0, 1)$ . Then

$$(a_1, a_2, \dots) \mapsto \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots$$

is a 1-1 correspondence between  $\Sigma_{\beta; \theta_0 \dots \theta_{q-1}}$  and  $C_{\beta; \theta_0 \dots \theta_{q-1}}$ .

Given  $x \in C_{\beta; \theta_0 \dots \theta_{q-1}}$ , assume that  $x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots$  for some  $(a_1, a_2, \dots) \in \Sigma_{\beta; \theta_0 \dots \theta_{q-1}}$ . Define  $(i_1, i_2, \dots) \in \Sigma_\alpha$  according to  $a_k = \theta_{i_k}$ . Then  $x = \lim_{k \rightarrow \infty} x_k$  where

$$x_k = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots + \frac{a_k}{\beta^k} = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}(0).$$

On the other hand, for any  $(i_1, i_2, \dots) \in \Sigma_\alpha$  and any  $k > 0$  we have  $(\theta_{i_1}, \theta_{i_2}, \dots, \theta_{i_k}, \theta_{i_0}, \dots) \in \Sigma_{\beta; \theta_0 \dots \theta_{q-1}}$ . Denote  $x^* = \frac{\theta_0}{\beta} + \frac{\theta_0}{\beta^2} + \dots$ . Then

$$\begin{aligned} x_k &= f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}(x^*) \\ &= \frac{\theta_{i_1}}{\beta} + \frac{\theta_{i_2}}{\beta^2} + \dots + \frac{\theta_{i_k}}{\beta^k} + \frac{\theta_0}{\beta^{k+1}} + \dots \in C_{\beta; \theta_0 \dots \theta_{q-1}}. \end{aligned}$$

Hence  $\lim_{k \rightarrow \infty} f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}(x^*) \in C_{\beta; \theta_0 \dots \theta_{q-1}}$ . This shows that  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is the  $\alpha$ -attractor of the IFS  $\left([0, \frac{\beta}{\beta-1}]; f_0, \dots, f_{q-1}\right)$ .

If we have  $f_i(C_{\beta; \theta_0 \dots \theta_{q-1}}) \cap f_j(C_{\beta; \theta_0 \dots \theta_{q-1}}) = \emptyset$  for  $i \neq j$  then Theorem 2 can be obtained by Theorem 1. Unfortunately in some cases we do not have the separation condition. For example, when  $\beta = 4$ , the separation condition does not hold for  $C_{4; 013}$ . For Hausdorff dimension we may use the following argument.

For  $\alpha' < \alpha$ , we use  $C_{\beta; \theta_0 \dots \theta_{q-1}}^{\alpha'}$  to denote the  $\alpha'$ -attractor of the IFS in consideration. Then  $C_{\beta; \theta_0 \dots \theta_{q-1}}^{\alpha'} \subset C_{\beta; \theta_0 \dots \theta_{q-1}}$ . Since  $\alpha' < \alpha$ , we have  $1 \notin C_{\beta; \theta_0 \dots \theta_{q-1}}^{\alpha'}$ . Then the separation condition holds for  $C_{\beta; \theta_0 \dots \theta_{q-1}}^{\alpha'}$ . Therefore,

$$\dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}}) \geq \dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}}^{\alpha'}) = \frac{\log \alpha'}{\log \beta}$$

for any  $\alpha' < \alpha$ . This implies that

$$\dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}}) \geq \frac{\log \alpha}{\log \beta}.$$

However, this does not supply any information about the Hausdorff measure of dimension  $\frac{\log \alpha}{\log \beta}$ . We will use similar discussion as the proof of Theorem 1.

Notice that when separation condition fails we must have  $0, 1 \in C_{\beta; \theta_0 \dots \theta_{q-1}}$ . For an  $\alpha$ -admissible sequence  $(i_1, \dots, i_k)$  let  $I_{i_1 \dots i_k}$  to denote the smallest closed interval which contains all numbers whose  $\beta$ -expansion starting with  $(\theta_{i_1}, \dots, \theta_{i_k})$ . If  $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$  then  $I_{i_1 \dots i_k} \cap I_{j_1 \dots j_k}$  contains at most one point. Let  $\mathcal{U} = \{U_1, \dots, U_n\}$  be a finite cover of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$ . If  $\beta^{-k-1} < |U_i| \leq \beta^{-k}$  then  $U_i$  intersects at most three  $I_{i_1 \dots i_k}$ . Then

$$\sum_{I_{i_1 \dots i_k} \cap U_i \neq \emptyset} |I_{i_1 \dots i_k}|^s \leq 3\beta^{-ks} < 3\beta^s |U_i|^s$$

Assume that  $|U_i| > \beta^{-l}$  for all  $i$ . By Lemma 1  $(i_1, \dots, i_k)$  can be extend to at most  $\frac{\alpha^{l-k+1}}{\alpha-1}$  many  $(i_1, \dots, i_k, \dots, i_l)$ . Then

$$\sum_{I_{i_1 \dots i_l} \cap U_i \neq \emptyset} |I_{i_1 \dots i_l}|^s \leq \frac{3\alpha^{l-k+1}}{\alpha-1} \beta^{-ls} = \frac{3\alpha^{-k+1}}{\alpha-1} < \frac{3\alpha^2 |U_i|^s}{\alpha-1}$$

where  $s = \frac{\log \alpha}{\log \beta}$ . Hence

$$\sum_{i=1}^N |U_i|^s > \frac{\alpha - 1}{3\alpha} \sum_{(i_1, \dots, i_l)} |I_{i_1 \dots i_l}|^s \quad (10)$$

where the sum is over all  $\alpha$ -admissible  $(i_1, \dots, i_l)$ .

We claim that if the separation condition does not hold then we have  $\alpha > 2$ . In fact if  $\alpha \leq 2$  we have the following three possibilities:

1.  $\theta_1 < \lfloor \beta \rfloor$ , which implies  $1 \notin C_{\beta; \theta_0 \dots \theta_{q-1}}$ ;
2.  $\theta_0 > 0$ , which implies  $0 \notin C_{\beta; \theta_0 \dots \theta_{q-1}}$ ;
3.  $\theta_1 \geq \theta_0 + 1$ .

In all these three cases we have  $f_0(C_{\beta; \theta_0 \dots \theta_{q-1}}) \cap f_1(CC) = \emptyset$ .

Now we estimate the number of  $I_{i_1 \dots i_l}$  that  $|I_{i_1 \dots i_l}| = \beta^{-l}$ . Let

$$A_k = \{(i_1, \dots, i_k) | (i_1, \dots, i_{k-1}, i_k + 1) \text{ is } \alpha\text{-admissible}\}.$$

If  $(i_1, \dots, i_k) \in A_k$  then  $(\theta_{i_1}, \dots, \theta_{i_{k-1}}, \theta_{i_k+1})$  is  $\beta$ -admissible. Thus  $(\theta_{i_1}, \dots, \theta_{i_{k-1}}, \theta_{i_k} + 1)$  is  $\beta$ -admissible since  $\theta_{i_k} + 1 \leq \theta_{i_k+1}$ . Therefore we have  $|I_{i_1 \dots i_k}| = \beta^{-k}$ . Let  $\mathcal{S}_\alpha^k$  of all  $\alpha$ -admissible  $(i_1, \dots, i_k)$ . By Lemma 1 we have  $|\mathcal{S}_\alpha^k| \geq \beta^k$ . We show that  $|A_k| \geq c|\mathcal{S}_\alpha^k|$  for some constant  $c$ .

Let  $B_k = \mathcal{S}_\alpha^k \setminus A_k$ . We estimate  $\frac{|B_k|}{|A_k|}$ . Notice that  $|B_{k+1}| \leq |\mathcal{S}_\alpha^k| = |A_k| + |B_k|$  and  $|A_{k+1}| \geq \lfloor \alpha \rfloor \cdot |A_k|$ . Then

$$\frac{|B_{k+1}|}{|A_{k+1}|} \leq \frac{|A_k| + |B_k|}{\lfloor \alpha \rfloor \cdot |A_k|} = \frac{1}{\lfloor \alpha \rfloor} \left( 1 + \frac{|B_k|}{|A_k|} \right).$$

Continue this discussion we get that for any  $k$  we have

$$\frac{|B_k|}{|A_k|} \leq \frac{1}{\lfloor \alpha \rfloor} + \frac{1}{\lfloor \alpha \rfloor^2} + \dots = \frac{1}{\lfloor \alpha \rfloor - 1}.$$

Therefore,

$$\frac{|A_l|}{|\mathcal{S}_\alpha^l|} \geq \frac{1}{1 + \frac{1}{\lfloor \alpha \rfloor - 1}} = \frac{\lfloor \alpha \rfloor - 1}{\lfloor \alpha \rfloor}.$$

By this and (10) we obtain

$$\begin{aligned} \sum_{i=1}^N |U_i|^s &> \frac{\alpha - 1}{3\alpha} \sum_{(i_1, \dots, i_l) \in A_l} |I_{i_1 \dots i_l}|^s \\ &\geq \frac{\alpha - 1}{3\alpha} \cdot \frac{\lfloor \alpha \rfloor - 1}{\lfloor \alpha \rfloor} \cdot |\mathcal{S}_\alpha^l| \cdot \beta^{-sl} \\ &\geq \frac{\alpha - 1}{3\alpha} \cdot \frac{\lfloor \alpha \rfloor - 1}{\lfloor \alpha \rfloor} \cdot \alpha^l \beta^{-sl} = \frac{\alpha - 1}{3\alpha} \cdot \frac{\lfloor \alpha \rfloor - 1}{\lfloor \alpha \rfloor} \end{aligned}$$

for all finite cover  $\mathcal{U}$ , where  $s = \frac{\log \alpha}{\log \beta}$ . This proves that the  $s$ -dimensional Hausdorff measure of  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is positive.  $\square$

**Remark.** Observe that the Hausdorff dimension of  $C_{\beta; \theta_0 \dots \theta_q}$  is related to “base change”. For the classical Cantor set  $C$ , if we change base 3 into base 2 and the digit 2 into 1, we get an (almost) 1-1 map from  $C$  to  $[0, 1]$ . Theorem 2 demonstrates that the “base change” property also holds for non-integer bases.

**Example.** Let  $\beta_1, \beta_2$  be given by  $1 = \frac{3}{\beta_1} + \frac{2}{\beta_1^2}$  and  $1 = \frac{3}{\beta_2} + \frac{3}{\beta_2^2}$ . Then  $\beta_1 = \frac{3+\sqrt{17}}{2}$  and  $\beta_2 = \frac{3+\sqrt{21}}{2}$ . We consider  $C_{\beta; 013}$  for  $\beta \in [\beta_1, \beta_2]$ . It is easy to see that  $\Sigma_{\beta_1; 013} = \Sigma_{\beta_2; 013}$ , and hence  $\Sigma_{\beta; 013} = \Sigma_{\beta_1; 013}$  for any  $\beta \in [\beta_1, \beta_2]$ . The maximal sequence of  $\Sigma_{\beta_1; 013}$  is  $(3, 1, 3, 1, \dots)$ . Thus  $\alpha_{\beta; 013}$  is determined by

$$1 = \frac{2}{\alpha} + \frac{1}{\alpha^2} + \frac{2}{\alpha^3} + \frac{1}{\alpha^4} + \dots$$

which gives  $\alpha = 1 + \sqrt{3}$ . therefore, for any  $\beta \in [\beta_1, \beta_2]$

$$\dim_H(C_{\beta; 013}) = \frac{\log(1 + \sqrt{3})}{\log \beta}.$$

By this we see that  $\dim_H(C_{\beta; 013})$  is decreasing with respect to  $\beta$  on the interval  $[\beta_1, \beta_2]$ .

**Theorem 3.** For fixed  $\theta_0, \dots, \theta_{q-1}$ ,  $\dim_H(C_{\beta; \theta_0 \dots \theta_{q-1}})$  is continuous for  $\beta > \theta_{q-1}$ .

*Proof.* We need to prove that  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}}$  is continuous with respect to  $\beta$  for  $\beta > \theta_{q-1}$ . Obviously,  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}}$  is non-decreasing with respect to  $\beta$ . It is clear that we have  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}} > q - 1$ . If we can show that for any  $\alpha \in (q - 1, q]$ , there exists a  $\beta$  such that  $\alpha = \alpha_{\beta; \theta_0 \dots \theta_{q-1}}$  by the monotone property we see that it is continuous.

Given  $\alpha \in (q - 1, q]$ , assume that the  $\alpha$ -expansion of 1 is

$$1 = \frac{\epsilon_1}{\alpha} + \frac{\epsilon_2}{\alpha^2} + \dots$$

Let  $\beta$  be determined by

$$1 = \frac{\theta_{\epsilon_1}}{\beta} + \frac{\theta_{\epsilon_2}}{\beta^2} + \dots$$

Then we have  $\alpha = \alpha_{\beta; \theta_0 \dots \theta_{q-1}}$ . Therefore,  $\alpha_{\beta; \theta_0 \dots \theta_{q-1}}$  is, and in turn,  $C_{\beta; \theta_0 \dots \theta_{q-1}}$  is continuous with respect to  $\beta$ .  $\square$

Now let us have a closer look at  $C_{\beta; 013}$  for  $\beta \in (3, 4]$ . By Theorem 3 we know that  $\dim_H(C_{\beta; 013})$  is continuous with respect to  $\beta$ . We have  $\dim_H(C_{4; 013}) = \frac{\log 3}{\log 4}$  and

$$\lim_{\beta \downarrow 3} \dim_H(C_{\beta; 013}) = \frac{\log 2}{\log 3} = \dim_H(C_{3; 01}).$$

As stated in the above,  $\dim_H(C_{\beta;013})$  is decreasing for  $\beta \in \left(\frac{3+\sqrt{17}}{2}, \frac{3+\sqrt{21}}{2}\right)$ . In fact, if  $\beta_l$  and  $\beta_r$  be determined by

$$1 = \frac{3}{\beta_l} + \sum_{i=2}^m \frac{a_i}{\beta_l^i} + \frac{2}{\beta_l^{m+1}} \quad (11)$$

and

$$1 = \frac{3}{\beta_r} + \sum_{i=2}^m \frac{a_i}{\beta_r^i} + \frac{3}{\beta_r^{m+1}} \quad (12)$$

where  $a_i \in \{0, 1, 3\}$  with

$$(a_i, a_{i+1}, \dots, a_m, 3, 0, \dots) < (3, a_1, \dots, a_m, 3, 0, \dots),$$

then for any  $\beta \in [\beta_l, \beta_r]$  we have the same value for  $\alpha_{\beta;013}$  which is given by

$$1 = \frac{2}{\alpha^k} + \sum_{i=2}^m \frac{b_i}{\alpha^i} + \frac{2}{\alpha^{m+1}} \quad (13)$$

where  $b_i = a_i$  if  $a_i \in \{0, 1\}$  or  $b_i = 2$  if  $a_i = 3$ .

Let  $V = [3, 4] \setminus \cup (\beta_l, \beta_r)$  where the union is for all possible  $\beta_l, \beta_r$  defined in the above.

**Theorem 4.**  $\beta \in V$  if and only if the  $\beta$ -expansion of 1 is

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots \quad (14)$$

or

$$1 = \frac{a_1}{\beta} + \dots + \frac{a_k}{\beta^k} + \frac{2}{\beta^{k+1}} \quad (15)$$

where  $a_i \in \{0, 1, 3\}$ . For  $\beta \in (3, 4) \setminus V$  we have

$$\frac{d \dim_{\beta}(C_{\beta;013})}{d\beta} < 0.$$

If  $\beta \in V$  but  $\beta \neq \beta_l$  or  $\beta_r$ ,

$$\frac{d \dim_H(C_{\beta;013})}{d\beta} = \infty.$$

If  $\beta = \beta_l$  or  $\beta_r$ , then the left sided or right sided derivative of  $\dim_H(C_{\beta;013})$  with respect to  $\beta$  is  $\infty$ .

*Proof.* Given  $\beta \in (3, 4)$ , if the  $\beta$ -expansion of 1 contains digit 2, then  $\beta \in [\beta_l, \beta_r]$  for some pair of  $\beta_l, \beta_r$  defined in the above. Hence we have either  $\beta \notin V$  or  $\beta = \beta_l$ . Therefore,  $\beta \in V$  implies (14) or (15) and vice versa.

Since for  $\beta \in [\beta_1, \beta_2]$  we have the same value of  $\alpha_{\beta;013}$ , we have  $\frac{d \dim_{\mathcal{G}}(C_{\beta;013})}{d\beta} < 0$  for  $\beta \in (3, 4) \setminus V$ .

Taking  $\beta \in V$ , we assume that  $\beta$  is non-simple and show that  $\lim_{\beta' \downarrow \beta} \frac{\alpha - \alpha'}{\beta - \beta'} = \infty$ . For other cases the discussion is similar and simpler. Let

$$1 = \frac{a_1}{\beta} + \frac{a_2}{\beta} + \dots$$

where  $a_i \in \{0, 1, 3\}$  and  $a_i \neq 0$  for infinitely many  $i$ . Given  $k \geq 1$ , choose  $i_k > k$  such that  $a_{i_k} \neq 3$ . Let  $\beta_k$  be given by

$$1 = \sum_{i=1}^{i_k-1} \frac{a_i}{\beta_k^i} + \frac{1}{\beta_k^{i_k-1}} \left( \frac{a_1}{\beta_k} + \frac{a_2}{\beta_k^2} + \dots \right).$$

Use  $\alpha$  and  $\alpha_k$  to denote  $\alpha_{\beta;013}$  and  $\alpha_{\beta_k;013}$ . Then

$$1 = \frac{b_1}{\alpha} + \frac{b_2}{\alpha^2} + \dots$$

and

$$1 = \sum_{i=1}^{i_k-1} \frac{b_i}{\alpha_k^i} + \frac{1}{\alpha_k^{i_k-1}} \left( \frac{b_1}{\alpha_k} + \frac{b_2}{\alpha_k^2} + \dots \right).$$

where  $b_i = a_i$  if  $a_i = 0$  or  $1$  and  $b_i = 2$  if  $a_i = 3$ . We estimate  $\beta_k - \beta$  and  $\alpha_k - \alpha$ .

$$\begin{aligned} & \beta_k - \beta \\ &= \sum_{i=2}^{i_k-1} \left( \frac{a_i}{\beta_k^i} - \frac{a_i}{\beta^i} \right) + \left( \frac{a_1}{\beta_k^{i_k}} - \frac{a_{i_k}}{\beta^{i_k}} \right) + \left( \frac{a_2}{\beta_k^{i_k+1}} - \frac{a_{i_k+1}}{\beta^{i_k+1}} \right) + \dots \\ &\leq \frac{a_1 - a_{i_k}}{\beta_k^{i_k}} + \dots < \frac{c}{\beta^{i_k}} \end{aligned}$$

for some constant  $c$ .

$$\begin{aligned} & \alpha_k^{i_k} - \alpha^{i_k} \\ &= \sum_{i=1}^{i_k-1} b_i (\alpha_k^{i_k-i} - \alpha^{i_k-i}) + (b_1 - b_{i_k}) + T_{\alpha_k} y - T_{\alpha}^{i_k} 1 \end{aligned}$$

where  $y = \frac{b_1}{\beta_k} + \frac{b_2}{\beta_k^2} + \dots$  and  $T_{\alpha}^{i_k} 1 = \frac{b_{i_k+1}}{\beta} + \frac{b_{i_k+2}}{\beta^2} + \dots$ . Noting that  $b_1 = a_1 - 1 = 2$  and  $b_{i_k} \leq 1$ ,

$$\alpha_k - \alpha > \frac{1 + T_{\alpha_k} y - T_{\alpha}^{i_k} 1}{\alpha_k^{i_k-1} + \alpha_k^{i_k-1} \alpha + \dots + \alpha^{i_k-1}} > \frac{T_{\alpha_k} y}{i_k \alpha_k^{i_k}}.$$

Since  $\alpha_k \rightarrow \alpha$  when  $k \rightarrow \infty$ , we have

$$T_{\alpha_k} y = \frac{b_2}{\beta_k} + \frac{b_3}{\beta_k^2} + \dots \rightarrow \frac{b_2}{\beta} + \frac{b_3}{\beta^2} = T_{\alpha} 1 > 0.$$



Hence

$$\frac{\alpha_k - \alpha}{\beta_k - \beta} > \frac{cT_{\alpha_k} 1}{i_k} \left( \frac{\alpha_k}{\beta_k} \right)^{i_k} \rightarrow \infty.$$

This shows that  $\limsup_{\beta' \downarrow \beta} \frac{\alpha_k - \alpha}{\beta_k - \beta} = \infty$ , which implies that  $\lim_{\beta' \downarrow \beta} \frac{\alpha_k - \alpha}{\beta_k - \beta}$  since  $\alpha_{\beta;013}$  is a monotone function of  $\beta$ .

Now we have shown that for  $\beta \in V$  if it is non-simple then the right sided derivative of  $\alpha_{\beta;013}$  with respect to  $\beta$  is  $\infty$ . By this we see that the right sided derivative of  $\dim_H(C_{\beta;013})$  with respect to  $\beta$  is  $\infty$ . By similar discussions we can obtain that

$$\frac{d \dim_H(C_{\beta;013})}{d\beta} = \infty, \quad \beta \in V, \text{ but } \beta \neq \beta_l \text{ or } \beta_r$$

and

$$\frac{d \dim_H(C_{\beta;013})}{d\beta^-} = \infty$$

if  $\beta = \beta_l$ ,

$$\frac{d \dim_H(C_{\beta;013})}{d\beta^+} = \infty$$

if  $\beta = \beta_r$ , where we use “ $-$ ” and “ $+$ ” to denote the left and right handed derivatives.  $\square$

**Remark.** Although Theorem 4 is stated and proved for  $C_{\beta;013}$ , we have similar result for  $C_{\beta;\theta_0 \dots \theta_{q-1}}$  in general. We omit the details of the statement and proof in general case.

## 5 View $C_{\beta;02}$ through Different Eyes.

In this section, we assume that  $2 < \beta \leq 3$ . Let  $F_\beta = \{[0, 1]; f_0 : [0, 1] \mapsto [0, 1], f_2 : [0, \beta - 2] \mapsto [0, 1]\}$  be the local IFS defined in Section 1, where  $f_0(x) = \frac{x}{\beta}$  and  $f_2(x) = \frac{x+2}{\beta}$ . We show that  $C_{\beta;02}$  is the unique invariant set of  $T_\beta$ , for all  $\beta$  except a countable set.

**Theorem 5.** *For all  $\beta \in (2, 3]$ ,  $C_{\beta;02}$  is an invariant set of  $T_\beta$ . Let  $Q$  be the set of  $\beta \in (2, 3]$  such that the  $\beta$ -expansion of 1 is*

$$1 = \frac{\epsilon_1}{\beta} + \dots + \frac{\epsilon_k}{\beta^k}$$

where  $\epsilon_i \in \{0, 2\}$ . For  $\beta \in (2, 3] \setminus Q$ ,  $C_{\beta;02}$  is the only invariant set of  $T_\beta$

*Proof.* First we prove that  $C_{\beta;02}$  is an invariant set of  $F_\beta$  by showing that

$$C_{\beta;02} = f_0(C_{\beta;02}) \cup f_2(C_{\beta;02} \cap [0, \beta - 2]). \quad (16)$$

The compactness of  $C_{\beta;02}$  ensures that  $f_0(C_{\beta;02}) \cup f_2(C_{\beta;02} \cap [0, \beta - 2])$  is compact. For any  $x = (0.a_1 a_2 \dots)_\beta$  with  $a_i \in \{0, 2\}$  for all  $i$ , we have  $T_\beta x =$

$(0.a_2a_3\cdots)_\beta \in C_{\beta;02}$ . Then  $x = f_0(T_\beta x) \in f_0(C_{\beta;02})$  if  $a_1 = 0$ , or  $T_\beta x \in [0, \beta - 2)$  and  $x = f_2(T_\beta x) \in f_2(C_{\beta;02} \cap [0, \beta - 2])$  if  $a_1 = 2$ . Hence  $f_0(C_{\beta;02}) \cup f_2(C_{\beta;02} \cap [0, \beta - 2])$  contains all  $x \in [0, 1)$  whose  $\beta$ -expansion has only 0 or 2. By the definition of  $C_{\beta;02}$  and the compactness of  $f_0(C_{\beta;02}) \cup f_2(C_{\beta;02} \cap [0, \beta - 2])$  we get  $C_{\beta;02} \subset f_0(C_{\beta;02}) \cup f_2(C_{\beta;02} \cap [0, \beta - 2])$ . On the other hand, it is easy to see that  $f_0(C_{\beta;02}) \subset C_{\beta;02}$  and  $f_2(C_{\beta;02} \cap [0, \beta - 2]) \subset C_{\beta;02}$ . Therefore, we have (16).

Next show that if  $\beta \in (2, 3] \setminus Q$  then  $C_{\beta;02}$  is the only invariant set of  $F_\beta$ . Let  $A$  be an invariant set of  $T_\beta$ . Then  $A$  is compact and

$$A = f_0(A) \cup f_2(A \cap [0, \beta - 2]). \quad (17)$$

It is easy to see that  $0 \in A$ . By this and (17) we get that  $A$  contains all  $x$  whose  $\beta$ -expansion is finite with all entries equal to 0 or 2. Then we get  $C_{\beta;02} \subset A$ .

Assume that there exists  $x_0 \in A \setminus C_{\beta;02}$ . We claim that we have  $1 \in A$ . If  $x_0 \neq 1$  then  $x_0 = (0.a_1a_2\cdots)_\beta$  with  $a_k = 1$  for some  $k$ . Let  $k$  be the smallest integer with  $a_k = 1$ . By (17) we have either  $x_0 \in f_0(A)$  or  $x_0 \in f_2(A \cap [0, \beta - 2])$ . If  $x_0 \in f_0(A)$ , then  $y_0 = \beta x_0 \in A$ . This indicates that  $a_1 = 0$  or  $x_0 = \frac{1}{\beta}$ . If  $x_0 \in f_2(A \cap [0, \beta - 2])$ , then  $y_0 = \beta x_0 - 2 \in A$ . Hence we always have  $T_\beta x_0 \in A$ . Continue this discussion, we get  $T_\beta^{k-1} x_0 = (0.a_k a_{k+1} \cdots)_\beta \in A$ . Because  $a_k = 1$  we have  $T_\beta^{k-1} x_0 < \frac{2}{\beta}$ . Then  $T_\beta^{k-1} x_0 \notin f_2(A \cap [0, \beta - 2])$ . If  $T_\beta^{k-1} x_0 > \frac{1}{\beta}$ , then it is not contained in  $f_0(A)$ . Hence we must have  $T_\beta^{k-1} x_0 = \frac{1}{\beta}$ . By  $\frac{1}{\beta} \in A$  we get  $1 \in A$ .

If  $1 \in A$  then, by the above discussion,  $T^k 1 \in A$  for any  $k \geq 0$ . By (17) we see that either  $\beta$ -expansion of 1 does not contain 1 or it has an only 1 as the last non-zero term. However, if  $\beta$  is non-simple and the  $\beta$ -expansion of 1 does not contain 1, or  $\beta$  is simple with the only digit 1 as the last non-zero term, then we have  $1 \in C_{\beta;02}$  which implies  $x_0 \in C_{\beta;02}$ . Hence only when  $\beta$  is simple and the  $\beta$ -expansion of does not contain 1, that is,  $\beta \in Q$ , we have  $1 \in A \setminus C_{\beta;02}$ . Therefore,  $C_{\beta;02}$  is the only invariant set of  $T_\beta$  for  $\beta \in (2, 3] \setminus Q$ .  $\square$

Construct a compact set  $B_{\beta;02}$  by a interval removal process similar to that for the classical Cantor set. Firstly we remove the interval  $(\frac{1}{\beta}, \frac{2}{\beta})$  from the unit interval  $[0, 1]$ . Next we remove  $(\frac{1}{\beta^2}, \frac{2}{\beta^2})$  from the remaining interval  $[0, \frac{1}{\beta}]$  and remove  $(\frac{2}{\beta} + \frac{1}{\beta^2}, \frac{2}{\beta} + \frac{2}{\beta^2}) \cap [\frac{2}{\beta}, 1]$ . In general, at step  $n$  if a remaining interval  $[u_n, v_n]$  has  $v_n - u_n > \frac{1}{\beta^n}$ , we remove  $(u_n + \frac{1}{\beta^n}, u_n + \frac{2}{\beta^n}) \cap [u_n, v_n]$  from it.

**Theorem 6.**  $B_{\beta;02}$  is an invariant set of  $T_\beta$  for all  $\beta \in (2, 3]$ . For  $\beta \in Q$ , we have  $B_{\beta;02} \neq C_{\beta;02}$  and  $B_{\beta;02} \setminus C_{\beta;02}$  is a countable set of isolated points.

*Proof.* First we show that  $B_{\beta;02} = C_{\beta;02}$  for  $\beta \in (2, 3] \setminus Q$ . It is clear that  $B_{\beta;02} \supset C_{\beta;02}$ . By the construction process, if the  $\beta$ -expansion of a number has a 1 and at least one non-zero term after the digit 1 then it will be removed at some stage. Hence  $B_{\beta;02}$  consists of points in  $C_{\beta;02}$  and possibly the numbers

whose  $\beta$ -expansion is finite with an only digit 1 as the last non-zero term. As discussed above, when  $\beta \in (2, 3] \setminus Q$  these numbers are contained in  $C_{\beta;02}$ . Hence we have  $B_{\beta;02} = C_{\beta;02}$  for  $\beta \in (2, 3] \setminus Q$ .

Next we show that for  $\beta \in Q$  we have  $B_{\beta;02} \neq C_{\beta;02}$  and  $B_{\beta;02}$  is the only invariant set of  $T_\beta$  other than  $C_{\beta;02}$ . Assume that  $1 = (0.\epsilon_1\epsilon_2\cdots\epsilon_{k-1}2)_\beta$ . By the construction of  $B_{\beta;02}$ , the interval  $\left(\frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_{k-1}}{\beta^{k-1}} + \frac{1}{\beta^k}, \frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_{k-1}}{\beta^{k-1}} + \frac{2}{\beta^k}\right)$  is removed and 1 remains in  $B_{\beta;02}$ . Hence  $B_{\beta;02} \neq C_{\beta;02}$ . It is easy to see that in this case  $B_{\beta;02}$  contains all  $x = (0.a-1\cdots a_{k-1}1)_\beta$  where  $a_i \in \{0, 2\}$ . Therefore  $B_{\beta;02}$  is just the invariant set  $A$  in the proof of (5) when  $A \neq C_{\beta;02}$ . Hence  $B_{\beta;02}$  is an invariant set of  $T_\beta$  and  $T_\beta$  does not have invariant set other than  $B_{\beta;02}$  and  $C_{\beta;02}$ . It is obvious that  $B_{\beta;02} \setminus C_{\beta;02}$  consists of countably many isolated points.  $\square$

We can also define a normal IFS  $([0, 1]; f_0, f_2)$  by extending  $f_2$  to the whole interval with

$$f_2(x) = \begin{cases} \frac{x+2}{\beta} & x \in [0, \beta-2], \\ 1 & x \in (\beta-2, 1]. \end{cases}$$

Then  $B_{\beta;02}$  is the attractor of this IFS. We omit the details.

## 6 Further Research.

1. In Theorem 1 we obtained the Hausdorff dimension for  $\beta$ -attractors when all the maps have the same contract ratio under a separation condition  $f_i(E_\beta) \cap f_j(E_\beta) \cap E_\beta = \emptyset$ , whenever  $i \neq j$ . We believe that the separation condition can be replaced by an open set condition. Comparing with the  $\beta$ -expansion of real numbers in  $[0, 1)$ , we may define a  $\beta$ -open set condition as follows.

**$\beta$ -Open Set Condition.** Let  $F = (X; f_0, f_1, \dots, f_{n-1})$  be an IFS where  $X$  is a compact subset of  $R^m$  and  $f_i$  is a Lipschitz map with Lipschitz constant  $r_i < 1$  for  $0 \leq i \leq n-1$ . Let  $n-1 < \beta \leq n$  and  $E_\beta$  is the  $\beta$ -attractor. We say  $F$  satisfies a  $\beta$ -open set condition if there exists an open set  $O$  with  $E_\beta \subset \bar{O}$  such that  $f_i(O) \subset O$  and  $f_i(O) \cap f_j(O) = \emptyset$  for  $0 \leq i < j \leq n-1$ .

Note that in the above definition we do not require  $f_{n-1}(O) \subset O$  but assume  $E_\beta \subset \bar{O}$  which is automatically true in the original version of open set condition. It is easy to see that  $C_{\beta;\theta_0\cdots\theta_{q-1}}$  satisfies a  $\beta$ -open set condition if we choose  $O = (0, 1)$ . When  $\beta = n$  is an integer the original version of open set condition implies the  $\beta$ -open set condition for the same open set  $O$ . When all  $f_i$ 's are similarities, a  $\beta$ -open set condition with  $\beta$  being an integer also implies the original version of open set condition possibly with a different open set  $O'$ .

**Conjecture.** Let  $(X; f_0, f_1, \dots, f_{n-1})$  be an IFS such that

$$d(f_i(x), f_i(y)) = rd(x, y)$$

for all  $x, y \in X$  and  $0 \leq i \leq n-1$ , where  $0 < r < 1$ . Let  $1 < \beta < n$  and  $E_\beta$  be the  $\beta$ -attractor. Under the  $\beta$ -open set condition, we have

$$\dim(E_\beta) = \frac{\log \beta}{-\log r}.$$

The Hausdorff measure of  $E_\beta$  in its dimension is positive and finite.

2. Given an IFS  $(X; f_0, \dots, f_{n-1})$  with attractor  $E$  and a probability vector  $(p_0, \dots, p_{n-1})$  with  $p_i > 0$ , for any Borel measure  $\mu$  on  $X$  we define

$$F(\mu)(B) = \sum_{i=0}^{n-1} p_i \mu(f_i^{-1}(B))$$

for any Borel subset  $B$ . Then there is a unique measure  $\nu$  such that  $F(\nu) = \nu$ .  $\nu$  has  $E$  as its support and for any  $\mu$  the sequence  $\{F^{(k)}(\mu)\}$  weakly converges to  $\nu$ .

It is natural to study the invariant measure for  $\beta$ -attractors.

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